Stability of a Panel in Incompressible, Unsteady Flow

R. E. Kelly*
Massachusetts Institute of Technology, Cambridge, Mass.

The stability of an infinite panel is investigated when the panel is immersed in an airstream whose velocity is composed in part of an unsteady component that oscillates periodically with respect to time. The analysis indicates that neutrally stable waves in the panel can undergo parametric amplification due to the flow oscillation and therefore grow exponentially with time. It is also shown that the unsteadiness has a destabilizing effect on the mode of instability which is a result of dissipation in the panel. Finally, the possibility of stabilizing a membrane in a steady flow by varying its tension periodically with respect to time is discussed.

Introduction

Whereas much research has been devoted to the problem of the stability of a panel when it is immersed in a steady flow, the effects of an unsteady flow on a panel have apparently not yet been ascertained, although McClure[1] has considered the effects of a time-averaged, turbulent boundary layer. Yet one can easily imagine circumstances in which a panel might be subjected to severe unsteady loads. To represent such a flow adequately, spatial as well as temporal variations should be represented in the model. In this paper, however, emphasis is placed on the role of unsteadiness, and the freestream is assumed to vary as

$$U(t) = U_0(1 + \epsilon \cos\omega t) \qquad |\epsilon| \ll 1 \tag{1}$$

It is hoped that spatial dependence will be included in the flow model at some later time.

The present analysis follows the approach of Miles[2] to panel flutter, in which he assumed a panel of infinite length. A progressive wave analysis resulted which resembled the classic Kelvin-Helmholtz problem concerning the stability of an interface between fluids of different densities and velocities. The basic flow is assumed to be parallel and steady in that problem. The present results ensued from an investigation by the author[3,12] of the Kelvin-Helmholtz problem under unsteady conditions.

The assumption of an infinite panel suffers from the fact that it excludes important aerodynamic effects associated with the edges of a finite panel. However, it is certainly not obvious that such effects would eliminate the mechanism of energy transfer discussed in this paper, which is wholly dependent upon the unsteady nature of the basic flow. Further, it should be noted that traveling waves have been found experimentally with long panels.[10]

The present mechanism of energy transfer is best introduced by noting that, if one fills a container partly with water and causes the container to oscillate vertically with twice the frequency of a given mode, one will notice waves corresponding to that mode forming at the surface of the water. This result was first noticed by Faraday,[4] but a complete analysis of the phenomenon was given by Benjamin and Ursell[5] only in 1954. A simple physical argument for the subharmonic response can be given as follows. Consider that the container is pushed upward whenever the wave is at its maximum deflection away from the planar condition. Then the restoring force will be greater than in the case of a neutrally stable wave in a container at rest, and the kinetic energy of the wave will be correspondingly greater. Now consider that the container is pulled downward just when the planar condition is reached. Then the effective gravitational force will be less than in the static case, and the relatively greater kinetic energy of the wave can only be reflected in an increased amplitude of deflection of the surface. As the cycle repeats, resonance will result. Note that the wave has half the frequency of the vertical oscillation. The result is a clear example of parametric amplification, which is associated with systems in which some parameter (here gravity), which when constant defines in part the natural frequencies, varies periodically with time. The theory of parametric amplification and other examples of interest are discussed in books by McLachlan[6] and Stoker.[7]

Here we note only that, in both the Kelvin-Helmholtz and panel flutter problems, the wave speed of a neutrally stable wave is defined in part by the freestream velocity. This occurs because the pressure caused by a wave opposes the effects of gravity, tension, and bending stiffness. Hence, we might expect that, when the velocity and, consequently, the wave speed are changing periodically with time, an effect similar to the one just discussed might occur.

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* Graduate Research Assistant, Department of Aeronautics and Astronautics; now Senior Staff Scientist, National Physical Laboratory, Teddington, Middlesex, England.

Equations of Motion

The panel, with its midplane situated at $y = 0$, is considered to be exposed to an airstream in the positive x direction on both of its sides so that the unperturbed state is plane. It is taken to have a bending stiffness D per unit breath, a tension load T, an areal mass density m, and a coefficient of damping K. The equation relating the transverse deflection of the panel η to the pressure induced by the deflection is given as

$$m\frac{\partial^2\eta}{\partial t^2} + D\frac{\partial^4\eta}{\partial x^4} - T\frac{\partial^2\eta}{\partial x^2} + K\frac{\partial\eta}{\partial t} = \delta p_1 - \delta p_2 \tag{2}$$

where the subscripts $(\ )_1$ and $(\ )_2$ denote, respectively, the lower and upper surfaces of the panel.

The flow is irrotational, and so one may write an integral of the equations of motion for the air flow as

$$P/\rho_0 + \tfrac{1}{2}(u^2 + v^2) + \partial\phi/\partial t = F(t) \tag{3}$$

where ρ_0 is the air density, P the pressure, u and v the velocity components parallel and normal to the midplane, $F(t)$ an arbitrary function of time, and ϕ a velocity potential defined by

$$(\partial/\partial x, \partial/\partial y)\phi = u, v \tag{4}$$

We consider the basic flow, given by (1), to be perturbed by a disturbance of such small magnitude that the linear version of the relation (3) may be used for the disturbance pressure δp:

$$-\delta p = \rho_0 U \delta u + \rho_0 \partial\delta\phi/\partial t \tag{5}$$

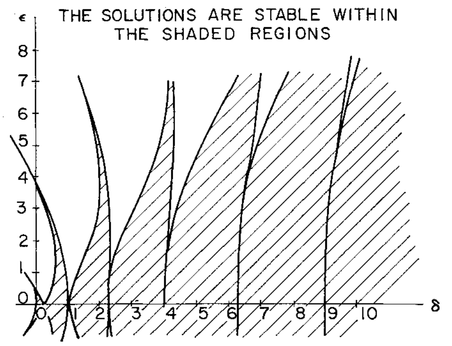

Fig. 1 The stability boundaries for Mathieu's equation $d^2y/d\tau^2 + [\delta + \epsilon \cos\tau]y = 0$.

The equation of continuity requires that the perturbation potential $\delta\phi$ satisfies

$$\partial^2\delta\phi/\partial x^2 + \partial^2\delta\phi/\partial y^2 = 0 \tag{6}$$

We assume that the perturbation can be expressed in terms of its Fourier components in the x direction and, accordingly, seek a solution where any function $f(x,y,t)$ has the following form:

$$f(x,y,t) = \bar{f}(y,t)e^{ikx} \tag{7}$$

where k is the wave number. Then the equation for $\delta\bar{\phi}$ becomes

$$\partial^2\delta\bar{\phi}/\partial y^2 - k^2\delta\bar{\phi} = 0 \tag{8}$$

so that

$$\delta\bar{\phi}_2 = A_2(t)e^{-ky} \qquad \delta\bar{\phi}_1 = A_1(t)e^{+ky} \tag{9}$$

The equation relating the displacement of the interface to the normal component of velocity is

$$\frac{\partial\eta}{\partial t} + U(t)\frac{\partial\eta}{\partial x} = \frac{\partial}{\partial y}\delta\phi_j\Big|_{z=0,\ j=1,2} \tag{10}$$

or

$$d\bar{\eta}/dt + ikU(t)\bar{\eta} = (-1)^{j+1}kA_j(t) \tag{11}$$

In order that the relation be unique, we require that

$$A_1(t) = -A_2(t) = -A(t) \tag{12}$$

so that Eq. (11) becomes

$$d\bar{\eta}/dt + ikU(t)\bar{\eta} = -kA \tag{13}$$

We may also use Eqs. (12, 9, and 5) to write Eq. (2) as

$$m\frac{d^2\bar{\eta}}{dt^2} + K\frac{d\bar{\eta}}{dt} + (Tk^2 + Dk^4)\bar{\eta} = 2ik\rho_0 uA + 2\rho_0\frac{dA}{dt} \tag{14}$$

Finally, Eqs. (13) and (14) may be combined to give the following single equation for $\bar{\eta}(t)$:

$$(m + 2\rho_0/k)d^2\bar{\eta}/dt^2 + (4i\rho_0 U + K)d\bar{\eta}/dt + [Tk^2 + Dk^4 + 2i\rho_0(dU/dt) - 2\rho_0 kU^2]\bar{\eta} = 0 \tag{15}$$

We will find it convenient to eliminate the first derivative term by means of the substitution

$$\bar{\eta}(t) = \hat{\eta}(t)\exp\left\{\frac{-Kt}{2(m + 2\rho_0/k)} - \frac{2i\rho_0}{(m + 2\rho_0/k)}\int U dt\right\} \tag{16}$$

to give the following alternative equation for $\hat{\eta}(t)$:

$$\frac{d^2\hat{\eta}}{dt^2} + \left[\frac{Tk^2 + Dk^4}{m + 2\rho_0/k} - \frac{2\rho_0 mkU^2}{(m + 2\rho_0/k)^2} - \frac{K^2}{4(m + 2\rho_0/k)^2} - \frac{2i\rho_0 UK}{(m + 2\rho_0/k)^2}\right]\hat{\eta} = 0 \tag{17}$$

Panel Without Dissipation

In the case $K = 0$, the equation governing the stability becomes

$$\frac{d^2\hat{\eta}}{dt^2} + \left[\frac{Tk^2 + Dk^4}{m + 2\rho_0/k} - \frac{2\rho_0 mkU^2}{(m + 2\rho_0/k)^2}\right]\hat{\eta} = 0 \tag{18}$$

Using the same notation as Miles, we define the following quantities

$$c_0^2 = (T + k^2D)/m \qquad \zeta_0 = c_0/U_0 \qquad \mu = 2\rho_0/mk \tag{19}$$

In the case of a steady flow, when $U(t) = U_0$, the panel is unstable when

$$\zeta_0^2 \leq \mu/(1 + \mu) \tag{20}$$

which agrees with the result (4.5) in the paper by Miles. The critical windspeed is therefore always greater than the wavespeed of the free oscillations for finite μ.

We now consider the case when the freestream has the functional dependence upon time as given in (1). Then Eq. (18) becomes

$$\frac{d^2\hat{\eta}}{dt^2} + \left[\frac{k^2c_0^2}{1 + \mu} - \frac{\mu k^2U_0^2}{(1 + \mu)^2}(1 + 2\epsilon\cos\omega t + \epsilon^2\cos^2\omega t)\right]\hat{\eta} = 0 \tag{21}$$

The case of greatest interest occurs for $|\epsilon| \ll 1$. Then Eq. (21) can be approximated by

$$\frac{d^2\hat{\eta}}{dt^2} + \left[\frac{k^2c_0^2}{1 + \mu} - \frac{\mu k^2U_0^2}{(1 + \mu)^2}(1 + 2\epsilon\cos\omega t)\right]\hat{\eta} = 0 \tag{22}$$

which has the form of Mathieu's equation

$$d^2y/d\tau^2 + [\delta + \epsilon\cos\tau]y = 0 \tag{23}$$

whose stability characteristics are shown in Fig. 1. Except for the subharmonic response ($\delta = \frac{1}{4}$), the regions of instability are all of order ϵ^2, and one would have to consider the full Eq. (21) in order to investigate these possibilities of instability. However, Eq. (22) is an adequate approximation to Eq. (21) for the subharmonic region, whose boundaries are defined for $|\epsilon| \ll 1$ by (see Stoker,[7] p. 212)

$$\delta = \tfrac{1}{4} \pm \epsilon/2 \tag{24}$$

Viscous effects tend to have a severe influence on the cuspidal shape of the other regions of stability, and usually only the subharmonic response is observed experimentally. Only the subharmonic region will be considered here. Then comparing Eq. (23) to Eq. (22), we may use Eq. (24) to write the condition for a subharmonic resonance as

$$\frac{k^2c_0^2}{1 + \mu} - \frac{\mu k^2U_0^2}{(1 + \mu)^2} = \frac{\omega^2}{4} \pm \frac{\epsilon\mu k^2U_0^2}{(1 + \mu)^2} \tag{25}$$

For small values of ϵ, the response is most likely when

$$\frac{(\zeta_0^2)_c}{1 + \mu} - \frac{\mu}{(1 + \mu)^2} = \left(\frac{\omega}{2kU_0}\right)^2 \tag{26}$$

or

$$(\zeta_0^2)_c = \frac{\mu}{1 + \mu} + (1 + \mu)\left(\frac{\omega}{2kU_0}\right)^2 \tag{27}$$

which is plotted in Fig. 2 for various values of the nondimensional frequency parameter. It should be realized that only below the curve $\omega/2kU_0 = 0$ is the system completely unstable (here we ignore the possibility that the system might be in one of the slender regions of stability which exist for $\delta < 0$ in Fig. 2). The other curves are ones of resonance, and instability will occur only if the characteristics of the system are such that they lie on one of the curves. We first notice that instability can occur when the mean value of the velocity is below the free-wave speed. The amount of reduction depends upon the wave number and frequency. For a fixed value of k, increasing the frequency tends to reduce the value of the mean speed that is required for resonance. Miles suggested that one might gain some information concerning a periodically supported panel from the analysis of an infinite panel by defining $k = n\pi/l$ where n is an integer and l is the distance between supports. Then from (25) for very small $|\epsilon|$, it is clear that the subharmonic response will not occur if

$$\omega^2 > 4(n\pi/l)^2[c_0^2/(1+\mu)] \tag{28}$$

This relation implies that only the higher harmonic modes will be stimulated by high-frequency oscillations of the stream.

When $\delta = \frac{1}{4}$, the solutions of Eq. (23) behave as $y \sim y_0 \exp\{\pm\epsilon\tau/2\}$. Hence, for the subharmonic condition given in Eq. (26), the solutions of Eq. (22) will behave as

$$\hat{\eta} \sim \hat{\eta}_0 \exp\left[\frac{2\epsilon\mu k^3 U_0^2}{(1+\mu)^2}\left\{\frac{c_0^2}{1+\mu} - \frac{\mu U_0^2}{(1+\mu)^2}\right\}^{-1/2} t\right] \tag{29}$$

Therefore the growth rate increases with the mean velocity and vanishes, of course, as $U_0 \to 0$. The growth rate also increases as the wavelength of the disturbance decreases.

Panel With Dissipation

It is now known that the presence of dissipation in a flexible surface can lead to a reduced value of the freestream velocity at which the surface becomes unstable. This seemingly paradoxical result has been discussed by Landahl,[8] Benjamin,[9] and, with special connection to panel flutter, by Dugundji, Dowell, and Perkin.[10]

Landahl found that, of the two wave speeds allowed by Eq. (18) for constant velocity flow, the wave that travels upstream for small U_0 causes the total kinetic and elastic energy of the system to decrease as its amplitude increases. Hence, any further reduction in the energy, such as through dissipation, will be reflected in an increased amplitude for this slow or class A wave. Because the oscillations provide an additional way in which the airstream can transmit energy to the panel, we now ask whether the oscillations might stabilize the class A waves by increasing their energy content.

The equation of interest is now Eq. (15). If we first take the case of a constant velocity $U(t) = U_0$, use the quantities defined in (19), and assume

$$\bar{\eta}(t) \sim e^{-i\sigma t} \tag{30}$$

then the characteristic equation becomes

$$-(1+\mu)\,\zeta^2 + (2\mu - iK_0)\zeta + (\zeta_0^2 - \mu) = 0 \tag{31}$$

where

$$\zeta = c/U_0 = \sigma/kU_0 \qquad K_0 = K/kmU_0 \tag{32}$$

Solving for ζ, we obtain

$$\zeta = \frac{\mu}{1+\mu} - \frac{iK_0}{2(1+\mu)} \pm \frac{1}{(1+\mu)}\left[\mu^2 - i\mu K_0 - \frac{K_0^2}{4} + (\zeta_0^2 - \mu)(1+\mu)\right]^{1/2} \tag{33}$$

When $\zeta_0^2 = \mu$, the two roots are

$$\zeta = 0 \qquad (2\mu - iK_0)/(1+\mu) \tag{34}$$

For $\zeta_0^2 < \mu$, one root has a positive imaginary part and therefore gives rise to growing solutions. This condition is more severe than that given in Eq. (20).

We now consider the velocity to be of the form given in (1) and concentrate on the case when ϵ and K_0 are small and of the same order of magnitude. From Eq. (33), it is clear that the condition for instability remains unchanged if we neglect terms of order $(K_0)^2$.

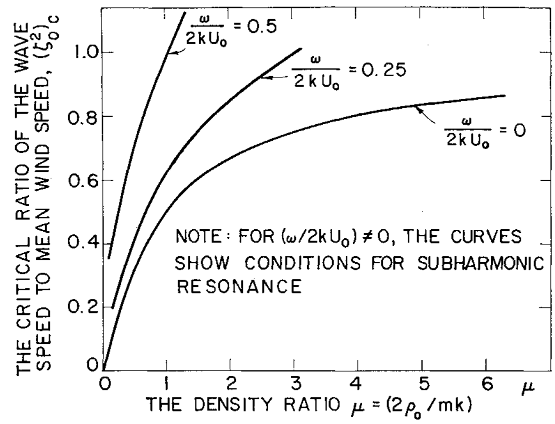

Fig. 2 The effect of airstream unsteadiness on the stability of a panel without damping in incompressible flow.

For this analysis, it is very convenient to apply the method of two scales, as presented by Kevorkian.[11] This method is particularly suitable to the present problem because two time scales exist, namely, the period of the wave and the scale for growth or decay of the wave, which is of order ϵ or K_0. We consider the case when the latter scale is much larger than the period of the wave. The method is illustrated by first applying it to the case of constant velocity flow.

We now define the two time variables

$$\bar{\tau} = kU_0 t \qquad \tau^* = K_0\bar{\tau} \qquad K_0 \ll 1 \tag{35}$$

and let $\hat{\eta} = \hat{\eta}(\bar{\tau}, \tau^*)$. The equation that replaces Eq. (17) is

$$\frac{\partial^2\hat{\eta}}{\partial\bar{\tau}^2} + 2K_0\frac{\partial^2\hat{\eta}}{\partial\bar{\tau}\partial\tau^*} + K_0^2\frac{\partial^2\hat{\eta}}{\partial\tau^{*2}} + \left[\lambda^2 - \frac{i\mu K_0}{(1+\mu)^2} - \frac{K_0^2}{4(1+\mu)^2}\right]\hat{\eta} = 0 \tag{36}$$

where

$$\lambda^2 = \zeta_0^2/(1+\mu) - \mu/(1+\mu)^2 \tag{37}$$

We now expand as

$$\hat{\eta}(\bar{\tau},\tau^*) = \hat{\eta}_0(\bar{\tau},\tau^*) + K_0\hat{\eta}_1(\bar{\tau},\tau^*) + \ldots \tag{38}$$

The solution for $\hat{\eta}_0(\bar{\tau}_1\,\tau^*)$ is

$$\hat{\eta}_0(\bar{\tau},\tau^*) = A(\tau^*)\cos\lambda\bar{\tau} + B(\tau^*)\sin\lambda\,\bar{\tau} \tag{39}$$

The equation for $\hat{\eta}_1(\bar{\tau},\tau^*)$ is then

$$\frac{\partial^2\hat{\eta}_1}{\partial\bar{\tau}^2} + \lambda^2\hat{\eta}_1 = \frac{i\mu}{(1+\mu)^2}\hat{\eta}_0 - 2\frac{\partial^2\hat{\eta}_0}{\partial\bar{\tau}\partial\tau^*} = \left[\frac{i\mu}{(1+\mu)^2}A - 2\lambda\frac{dB}{d\tau^*}\right]\cos\lambda\tau + \left[\frac{i\mu}{(1+\mu)^2}B + 2\lambda\frac{dA}{d\tau^*}\right]\sin\lambda\tau \tag{40}$$

In order to prevent secular terms from arising in the solution for $\hat{\eta}_1$, the terms in brackets in Eq. (40) must vanish. If we assume A, $B \sim \exp(\sigma\tau^*)$, then the characteristic equation for σ gives the result

$$\sigma = \pm\mu/2\lambda(1+\mu)^2 \tag{41}$$

For $\zeta_0^2 = \mu$, then $\lambda = \mu/(1+\mu)$, and we have

$$A,B \sim \exp\left\{\pm\frac{K_0\bar{\tau}}{2(1+\mu)}\right\} = \exp\left\{\pm\frac{Kt}{2(m+2\rho_0/k)}\right\} \tag{42}$$

Hence, for $\zeta_0^2 = \mu$, the growing solution exactly balances the exponentially decaying factor in Eq. (16), and we conclude that the panel is unstable for

$$\lambda < \mu/(1+\mu) \qquad \text{or} \qquad \zeta_0^2 < \mu \tag{43}$$

which agrees with the exact result.

We now apply the method to the unsteady problem, for which the governing equation is

$$\frac{1}{k^2U_0^2}\frac{d^2\hat{\eta}}{dt^2} + \left[\frac{\zeta_0^2}{1+\mu} - \frac{\mu}{(1+\mu)^2} - \frac{K_0^2}{4(1+\mu)^2} - \frac{i\mu K_0}{(1+\mu)^2} - \left\{\frac{2\mu}{(1+\mu)^2} + \frac{i\mu K_0}{(1+\mu)^2}\right\}\epsilon\cos\omega t - \frac{\mu\epsilon^2\cos^2\omega t}{(1+\mu)^2}\right]\hat{\eta} = 0 \quad (44)$$

When $|\epsilon| \ll K_0$, the result is that just shown to order K_0. If $K_0 \ll |\epsilon|$, the result is that shown in Eq. (25) to order ϵ. Here we are interested in the case when ϵ and K_0 are small but of the same order of magnitude. In order to include the limiting cases, we define

$$K_0 = K^*\beta \qquad \epsilon = \epsilon^*\beta \quad (45)$$

where β is the scale suitable to the particular problem. Thus, for $\epsilon^* \equiv 0$, we take $K^* = 1$ in order to obtain the steady results. We now define

$$\Lambda = kU_0/\omega \qquad \tau = \omega t \qquad \tau^* = \beta\tau \quad (46)$$

and let $\hat{\eta} = \hat{\eta}(\tau,\tau^*)$. The equation for $\hat{\eta}$ is

$$\frac{\partial^2\hat{\eta}}{\partial\tau^2} + 2\beta\frac{\partial^2\hat{\eta}}{\partial\tau\partial\tau^*} + \beta^2\frac{\partial^2\hat{\eta}}{\partial\tau^{*2}} + \Lambda^2\left[\lambda^2 - \frac{\beta\mu}{(1+\mu)^2}\{iK^* + 2\epsilon^*\cos\tau\} - \frac{\beta^2}{(1+\mu)^2}\left\{\frac{K^{*2}}{4} + iK^*\epsilon^*\cos\tau + \mu\epsilon^{*2}\cos^2\tau\right\}\right]\hat{\eta} = 0 \quad (47)$$

We now expand as

$$\hat{\eta}(\tau,\tau^*) = \hat{\eta}_0(\tau,\tau^*) + \beta\hat{\eta}_1(\tau,\tau^*) + \ldots \quad (48)$$

The solution for $\hat{\eta}_0$ is

$$\hat{\eta}_0(\tau,\tau^*) = A(\tau^*)\cos\Lambda\lambda\tau + B(\tau^*)\sin\Lambda\lambda\tau \quad (49)$$

and the equation for $\hat{\eta}_1$ is

$$\frac{\partial^2\hat{\eta}_1}{\partial\tau^2} + \Lambda^2\lambda^2\hat{\eta}_1 = -2\frac{\partial^2\hat{\eta}_0}{\partial\tau\partial\tau^*} + \frac{iK^*\mu\Lambda^2}{(1+\mu)^2}\hat{\eta}_0 + \frac{2\epsilon^*\mu\Lambda^2}{(1+\mu)^2}\cos\tau\,\hat{\eta}_0 \quad (50)$$

In general, the last term in Eq. (50) will not give rise to secular terms and therefore will not affect the steady stability conditions, as defined through the previous two terms. The exceptional case occurs for $\Lambda\lambda = \frac{1}{2}$ because then the product will give rise to terms with frequency of $\frac{1}{2}$. We consider now this condition, for which Eq. (50) becomes

$$\frac{\partial^2\hat{\eta}_1}{\partial\tau^2} + \frac{1}{4}\hat{\eta}_1 = \cos\frac{\tau}{2}\left[-\frac{dB}{d\tau^*} + \frac{iK^*\mu\Lambda^2}{(1+\mu)^2}A + \frac{\epsilon^*\mu\Lambda^2}{(1+\mu)^2}A\right] + \sin\frac{\tau}{2}\left[\frac{dA}{d\tau^*} + \frac{iK^*\mu\Lambda^2}{(1+\mu)^2}B - \frac{\epsilon^*\mu\Lambda^2}{(1+\mu)^2}B\right] + \frac{\epsilon^*\mu\Lambda^2}{(1+\mu)^2}\left(A\cos\frac{3\tau}{2} + B\sin\frac{3\tau}{2}\right) \quad (51)$$

In order to suppress secular terms, the terms in brackets again must vanish. If we again take $A, B \sim \exp(\sigma\tau^*)$, this condition is expressed by the determinant

$$\begin{vmatrix} -\sigma & \{\mu\Lambda^2/(1+\mu)^2\}(iK^* + \epsilon^*) \\ \{\mu\Lambda^2/(1+\mu)^2\}(iK^* - \epsilon^*) & \sigma \end{vmatrix} = 0$$

Upon expansion and solving for σ, we find

$$\sigma = \pm\frac{\mu\Lambda^2}{(1+\mu)^2}(K^{*2} + \epsilon^{*2})^{1/2} \quad (52)$$

or

$$A,B \sim \exp\left\{\pm\frac{\mu\Lambda^2}{(1+\mu)^2}(K^{*2} + \epsilon^{*2})^{1/2}\beta\omega t\right\} \quad (53)$$

Suppose that we first consider the effect of the oscillations on a class A wave. Then we let $K^* = 1$ and $K_0 = \beta$, and Eq. (53) becomes

$$A, B \sim \exp\left\{\pm\frac{\mu}{2\lambda(1+\mu)^2}(1 + \epsilon^{*2})^{1/2}\frac{Kt}{m}\right\} \quad (54)$$

The growing solution will predominate over the decaying component in Eq (16) when

$$\lambda < \mu(1 + \epsilon^{*2})^{1/2}/(1+\mu) \quad (55)$$

or

$$\zeta_0^2 < \mu + \mu^2\epsilon^{*2}/(1+\mu) \quad (56)$$

Thus, instability can occur at a lower value of U_0 when $\epsilon^* \neq 0$, and we conclude that the oscillations have a destabilizing effect on the class A waves, when $\lambda\Lambda = \frac{1}{2}$, which is contrary to our initial supposition. The critical value of ζ_0^2, so obtained from Eq. (56), is plotted as a function of μ and ϵ^* in Fig. 3. The condition $\lambda\Lambda = \frac{1}{2}$ may be used to write Eq. (55) as

$$\omega/2kU_0 < \mu(1 + \epsilon^{*2})^{1/2}/(1+\mu) \quad (57)$$

which indicates that high-frequency oscillations can only excite the short wavelength disturbances. Although it is not difficult to extend the analysis to include a region near $\lambda\Lambda = \frac{1}{2}$, one can show that this condition is most critical.

We might also ask how the presence of dissipation in the panel affects the subharmonic resonance. Now we let $\epsilon^* = 1$ and $\beta = \epsilon$, so that Eq. (53) becomes

$$A,B \sim \exp\left\{\pm\frac{\mu\Lambda^2}{(1+\mu)^2}(1 + K^{*2})^{1/2}\epsilon\omega t\right\} \quad (58)$$

It is convenient to rewrite Eq. (16) as

$$\bar{\eta}(t) = \hat{\eta}(t)\exp\left\{-\frac{|\epsilon|K^*\Lambda}{2(1+\mu)}\omega t - \frac{2i\rho_0}{(m + 2\rho_0/k)}\int U dt\right\} \quad (59)$$

Therefore the growing solution of Eq. (58) predominates when

$$[\mu\Lambda^2/(1+\mu)^2](1 + K^{*2})^{1/2} > K^*\Lambda/2(1+\mu) \quad (60)$$

or

$$\lambda < [\mu/(1+\mu)][(1 + K^{*2})^{1/2}/K^*] \quad (61)$$

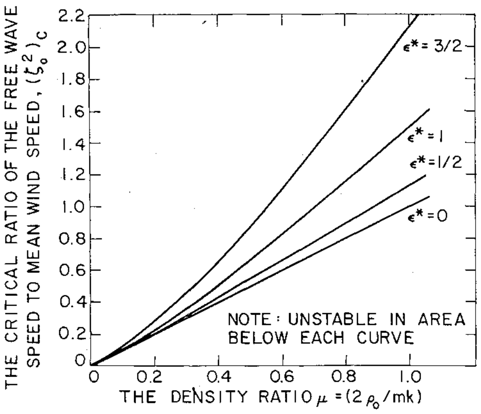

Fig. 3 The effect of airstream unsteadiness on the stability of a panel with damping in incompressible flow.

or

$$\zeta_0^2 < \frac{\mu}{1+\mu} + \frac{\mu^2}{1+\mu}\left(\frac{1+K^{*2}}{K^{*2}}\right) \quad (62)$$

It is clear that dissipation tends to have a strong stabilizing effect on the subharmonic response. For instance, as $K^* \to \infty$, the condition approaches the steady result. At any rate, the effect of nonzero K^* is to place a lower bound on permissable values of U_0 to cause instability. The critical value of ζ_0^2, as obtained from Eq. (62), is plotted as a function of K^* and μ in Fig. 4.

Membrane with Time-Varying Tension

Finally, it is interesting to speculate on whether the surface might be stabilized when the freestream is constant by allowing the tension to vary with time. Although such a suggestion is obviously impractical with regard to a metal panel, the possibility is worth investigating because use of a flexible surface has been suggested as a means of stabilizing boundary-layer flow. Under certain conditions, freestream unsteadiness might have a stabilizing influence; however, because one cannot select the frequencies desired, one must emphasize the destabilizing effects of freestream unsteadiness. On the other hand, one can visualize varying the surface tension in a predetermined manner and so possibly stabilize the surface.

We therefore consider the case when $D = 0$, $c_0^2 = T_0/m$, and $T = T_0(1 + \epsilon \cos\omega t)$. The equation for $\hat{\eta}(t)$ is now

$$\frac{d^2\hat{\eta}}{d\tau^2} + \left(\frac{kU_0}{\omega}\right)^2 \left[\frac{\zeta_0^2}{1+\mu} - \frac{\mu}{(1+\mu)^2} - \frac{K_0^2}{4(1+\mu)^2} - \frac{i\mu K_0}{(1+\mu)^2} + \frac{\zeta_0^2 \epsilon}{1+\mu}\cos\tau\right]\hat{\eta} = 0 \quad (63)$$

where $\tau = \omega t$. Consider first the case $K_0 = 0$, for which instability occurs when

$$\zeta_0^2 \leq \mu/(1+\mu) \quad (64)$$

With reference to Fig. 1, we see that the system can be stabilized for $\zeta_0^2 < \mu/(1+\mu)$ because Eq. (63) is exactly of the Mathieu type (for $K_0 = 0$). From Fig. 4 of the report by Kevorkian,[11] the boundaries of the region of stability for Mathieu's equation when $\delta < 0$ and $\epsilon \ll 1$ are

$$-\tfrac{1}{2}\epsilon^2 < \delta < \tfrac{1}{4} - \epsilon/2 - \epsilon^2/8 \quad (65)$$

For the present case, this condition can be expressed approximately as

$$\frac{2^{1/2}}{\zeta_0^2}\left(\frac{\omega}{ku_0}\right)[\mu - (1+\mu)\zeta_0^2]^{1/2} < |\epsilon| < \frac{1}{2}\left(\frac{\omega}{ku_0}\right)^2\left(\frac{1+\mu}{\zeta_0^2}\right) + 2\left[\frac{\mu}{(1+\mu)\zeta_0^2} - 1\right] \quad (66)$$

This relation can be simplified somewhat by defining

$$\zeta_0^2 = [\mu/(1+\mu)](1-\gamma) \quad (67)$$

so that γ is a measure of the extent to which the steady, critical windspeed can be exceeded. Then the relation (66) can be written as

$$2^{1/2}\left(\frac{\omega}{ku_0}\right)\left(\frac{1+\mu}{1-\gamma}\right)\left(\frac{\gamma}{\mu}\right)^{1/2} < |\epsilon| < \frac{1}{2}\left(\frac{\omega}{ku_0}\right)^2\frac{(1+\mu)^2}{\mu(1-\gamma)} + 2\left(\frac{\gamma}{1-\gamma}\right) \quad (68)$$

For most applications, μ is a small quantity ($\mu = 0.1 - 0.2$). Also, from practical considerations, we would like ϵ to be small. For $\epsilon \sim 0(\mu)$, it is clear from Eq. (68) that $(\omega/kU_0) \sim 0(\mu)$ in order to have $\gamma \sim 0(\mu)$. Hence, it would appear that

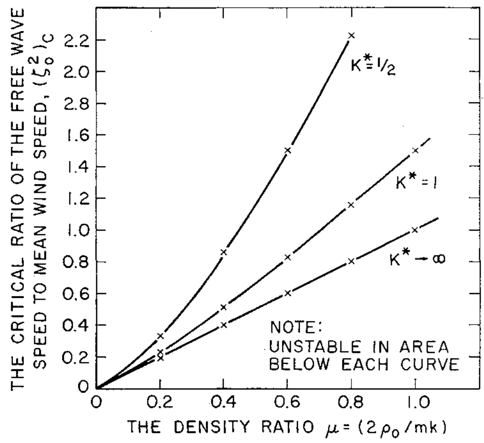

Fig. 4 The effect of panel damping on the subharmonic resonance.

a slight amount of stabilization can be achieved by suitably varying the tension at a relatively low frequency.

It should, however, be realized that, whereas the Kelvin-Helmholtz type instability can be postponed until $\zeta_0^2 < \mu/(1+\mu)$, a subharmonic instability is possible, once ω and k are fixed, for $\zeta_0^2 > \mu/(1+\mu)$ as given in Eq. (27). This instability will occur when

$$U_0^2 = \left(\frac{1+\mu}{\mu}\right)\left[c_0^2 - \left(\frac{\omega}{2k}\right)^2(1+\mu)\right] \quad (69)$$

One avoids this instability by simply not varying the tension until this condition is passed.

On the other hand, it does not seem possible to stabilize the class A wave by varying the tension when dissipation is important. By an analysis similar to that given in the preceding section, one finds that the oscillations have a destabilizing effect, similar to the effect of freestream unsteadiness.

Conclusions

The analysis indicates that a new mode of instability may exist for a panel when the panel is placed in a strongly fluctuating flow. The instability has a subharmonic character and results from parametric amplifications of waves that would be neutrally stable in a steady flow. Panel dissipation tends to stabilize this subharmonic resonance, although the oscillations have a destabilizing effect on the mode of instability due to panel damping. Finally, it would appear that a flexible surface in a steady flow can be stabilized somewhat by varying the tension with time if the panel dissipation is not too great.

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Elastic Stability of Thin-Walled Cylindrical and Conical Shells under Combined Internal Pressure and Axial Compression

V. I. Weingarten*
University of Southern California, Los Angeles, Calif.

E. J. Morgan†
Aerojet-General Corporation, Azusa, Calif.

and

Paul Seide‡
Aerospace Corporation, El Segundo, Calif.

The results of an extensive experimental program on the stability of cylindrical and conical shells under internal pressure and axial compression are presented. The use of the given data for design criteria is discussed, and recommendations are given. The experimental data for elastic, pressurized cylinders under axial compression indicated that the load carried by the cylinder in addition to that carried by internal pressure increases to the value given by small-deflection theory for unpressurized cylinders. The variation of the net load with internal pressure was found to depend on the radius-thickness ratio of the cylinder, and curves suitable for design were obtained. Experimental buckling loads obtained for conical shells were in good agreement with small-deflection theory predictions for sufficiently high pressures. The results indicated that the end-support condition may be more important for cones than cylinders. Insufficient data were obtained, however, to enable design curves to be recommended in the low-pressure region.

Nomenclature

E = Young's modulus of shell wall material
L = axial length of cylinder or cone
P = total axial load at buckling
p = uniform internal or external hydrostatic pressure
$\bar{p}$ = internal pressure parameter $\{p[(R_1/\cos\alpha)/t]^2/E$ for cones or $p/E(R/t)^2$ for cylinders$\}$
$\hat{p}$ = $p/E(R/t)^{5/3}$
R = cylinder radius
R_1 = radius of small end of cone
t = shell wall thickness
Z = cylinder curvature parameter $[3(1 - \nu^2)]^{1/2}L^2/Rt$
α = semivertex angle of cone
δ_{cr} = critical load shortening parameter $[(\Delta L_{cr}/L)R/t]$
γ = internal pressure parameter $\{[3(1 - \nu^2)/2]^{1/2}(p/E) \times [(R_1/\cos\alpha)/t]^2\}$
ϵ/ϵ_{cl} = ratio of compressive strain and classical buckling strain
ν = Poisson's ratio of wall material
σ, σ_{cr} = critical average compressive stress $P_{cr}/2\pi Rt$
$\bar{\sigma}_{cr}$ = critical average compressive stress coefficient $[\sigma_{cr}/(Et/R)]$
σ_c, σ_c = theoretical compressive buckling stress $\{E/[3(1 - \nu^2)]^{1/2}(t/R)\}$
$\bar{\sigma}_{\text{prop limit}}$ = stress coefficient at proportional limit of material $[\sigma_{\text{prop limit}}/(Et/R)]$
$\sigma_{\min}$ = minimum axial compressive stress in buckled state
$\bar{\sigma}_{\min}$ = minimum axial compressive stress coefficient $\{\sigma_{\min}/[E(t/R)]\}$
ΔL_{cr} = total end shortening at buckling
ζ = cone geometry parameter $\{[12(1 - \nu^2)]^{1/2} \times [(R_1/\cos\alpha)/t]\cot^2\alpha\}$

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* Assistant Professor, Civil Engineering. Member AIAA.
† Head, Aerospace Tankage Structures Support Section. Member AIAA.
‡ Staff Scientist. Associate Fellow Member AIAA.

Introduction

INCREASED emphasis has recently been placed upon the internally pressurized monocoque cylinders and cones as an efficient load carrying structure for missile applications. As a result, several experimental investigations have been reported in the literature.[1–5] Practically all the results obtained to date for cylindrical shells seem inconsistent with current explanations of cylinder behavior. With the exception of the study of Ref. 1, even large amounts of internal pressure did not stabilize the test cylinders to the extent that the theoretical